The Deligne-Simpson problem for zero index of rigidity *

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To the memory of my mother

Abstract

We consider the *Deligne-Simpson problem*: *Give necessary and sufficient conditions for the choice of the conjugacy classes $c_j \subset gl(n, \mathbf{C})$ or $C_j \subset GL(n, \mathbf{C})$, $j = 1, \ldots, p+1$, so that there exist irreducible $(p+1)$-tuples of matrices $A_j \in c_j$ whose sum is 0 or of matrices $M_j \in C_j$ whose product is I.* The matrices A_j (resp. M_j) are interepreted as matrices-residua of Fuchsian linear systems (resp. as monodromy operators of regular systems) on Riemann's sphere.

We consider the case when the sum of the dimensions of the conjugacy classes c_j or C_j is $2n^2$ and we prove a theorem of non-existence of such irreducible $(p+1)$-tuples.

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1 Introduction

In the present paper we sonsider a particular case of the *Deligne-Simpson problem (DSP)*:

Give necessary and sufficient conditions for the choice of the conjugacy classes $c_j \subset gl(n,\mathbf{C})$ or $C_j \subset GL(n,\mathbf{C})$, $j = 1,\ldots,p+1$, so that there exist irreducible $(p+1)$-tuples of matrices $A_j \in c_j$ or $M_j \in C_j$ satisfying respectively the equality

$$A_1 + \ldots + A_{p+1} = 0 \tag{1}$$

or

$$M_1 \ldots M_{p+1} = I \ . \tag{2}$$

"Irreducible" means "not having a common proper invariant subspace", i.e. impossible to conjugate simultaneously the $(p+1)$ matrices to a block upper-triangular form. The problem is connected with the theory of linear regular systems of differential equations on Riemann's sphere:

$$\dot{X} = A(t)X \tag{3}$$

Here the $n\times n$-matrix $A(t)$ is meromorphic on $\mathbf{C}P^1$, with poles at the points a_1, …, a_{p+1}; the unknown variables X form also a matrix $n \times n$. Such a system is called *regular* at the pole a_j if one has $||X(t-a_j)|| = O(|t-a_j|^{N_j})$ for some $N_j \in \mathbf{R}$ when the solution is restricted to a sector of sufficiently small radius and centered at a_j.

A particular case of a regular system is a *Fuchsian* one, i.e. with logarithmic poles:

$$\mathrm{d}X/\mathrm{d}t = (\sum_{j=1}^{p+1} A_j/(t-a_j))X \tag{4}$$

where $A_j \in gl(n,\mathbf{C})$ are its *matrices-residua*; in the absence of a pole at ∞ one has (1).

As a result of a linear change of variables

$$X \mapsto W(t)X \tag{5}$$

the matrix $A(t)$ of a regular system (3) undergoes the *gauge transformation*

$$A(t) \mapsto -W^{-1}\dot{W} + W^{-1}A(t)W \tag{6}$$

The $n \times n$-matrix W is meromorphic on $\mathbf{C}P^1$, its poles if any are usually among the points a_j, and outside them $\det W \not\equiv 0$. The only invariant of a regular system under the linear changes (5) is its *monodromy group*. This is the group generated by the *monodromy operators*.

A *monodromy operator* is a linear operator mapping the solution space of a regular system onto itself. It is defined as follows: one fixes a base point $a \neq a_j$ for $j = 1,\ldots,p+1$, the value at a of the solution X, i.e. a matrix $B \in GL(n,\mathbf{C})$ and a closed contour Γ passing through a. The

monodromy operator M defined by the homotopy equivalence class of the contour Γ maps the solution X with $X|_{t=a} = B$ onto the value at a of its analytic continuation along the contour (notation: $X \stackrel{\Gamma}{\mapsto} XM$).

Fix $(p+1)$ contours whose homotopy equivalence classes generate $\pi_1(\mathbf{C}P^1\backslash\{a_1,\ldots,a_{p+1}\})$. One usually chooses the contours such that Γ_j consists of a segment $[a,x_j]$ (x_j is close to a_j), of a small circumference (centered at a_j, passing through x_j, circumventing a_j counterclockwise and not containing inside any other pole a_i) and of the segment $[x_j,a]$. We assume that for $i\neq j$ one has $\Gamma_i\cap\Gamma_j=\{a\}$ and that the index of the contour increases when one turns around a clockwise. For such a choice of the contours the monodromy operators M_j satisfy the condition (2). This means that one can choose as generators of the monodromy group any p out of the $p+1$ operators M_j.

The monodromy group is an antirepresentation of $\pi_1(\mathbf{C}P^1\backslash\{a_1,\ldots,a_{p+1}\})$ into $GL(n,\mathbf{C})$ because one has $X \stackrel{\Gamma_i\Gamma_j}{\mapsto} XM_jM_i$ (although we often write "representation" instead). The change of a and B changes the monodromy group to a conjugate one.

If the contours defining the operators M_j are chosen like above, then M_j is conjugate to the corresponding *operator of local monodromy* defined by a small lace circumventing the pole a_j counterclockwise. Therefore in the case of matrices M_j the DSP admits the interpretation:

For which $(p+1)$-tuples of local monodromies do there exist irreducible monodromy groups with such local monodromies ?

Remark 1 The eigenvalues $\lambda_{k,j}$ of the matrix-residuum A_j of a Fuchsian system are connected with $\sigma_{k,j}$, the ones of the monodromy operator M_j by $\exp(2\pi i\lambda_{k,j})=\sigma_{k,j}$.

2 Definitions and known facts

2.1 The quantities d_j, r_j and κ; the construction Ψ; (poly)multiplicity vectors

Definition 2 A Jordan normal form (JNF) of size n is a collection of positive integers indexed by two indices – $J^n=\{b_{i,k}\}$ – where k is the index of an eigenvalue, i is the index of the Jordan block of size $b_{i,k}$ with this eigenvalue; $k=1,\ldots,\rho$, $i=1,\ldots,s_k$. We assume that all ρ eigenvalues are distinct and that for each k one has $b_{1,k}\geq\ldots\geq b_{s_k,k}$.

Convention. All Jordan matrices and Jordan blocks are presumed to be upper-triangular.

Definition 3 Denote by $J(X)$ the JNF of the matrix X. We say that the DSP is *solvable* (resp. *weakly solvable*) for a given $\{J_j^n\}$ and given eigenvalues if there exists an irreducible $(p+1)$-tuple (resp. a $(p+1)$-tuple with a trivial centralizer) of matrices M_j satisfying (2) or of matrices A_j satisfying (1), with $J(M_j)=J_j^n$ or $J(A_j)=J_j^n$ and with the given eigenvalues. By definition, the DSP is solvable for $n=1$.

For a given conjugacy class C (in $gl(n,\mathbf{C})$ or $GL(n,\mathbf{C})$) we denote by $d(C)$ its dimension (which is always even) and by $r(C)$ the quantity $\min_{\lambda\in\mathbf{C}}\operatorname{rk}(X-\lambda I)$ for $X\in C$. The quantity $n-r(C)$ is the greatest number of Jordan blocks with one and the same eigenvalue. We set $d_j=d(c_j)$ (resp. $d_j=d(C_j)$) and $r_j=r(c_j)$ (resp. $r_j=r(C_j)$). The quantities $r(C)$ and $d(C)$ depend not on the conjugacy class C but only on the JNF defined by it.

The following two conditions are necessary for the existence of irreducible $(p+1)$-tuples of matrices M_j satisfying (2) or of matrices A_j satisfying (1), see [Si] and [Ko3], [Ko4]:

$$d_1+\ldots+d_{p+1}\geq 2n^2-2 \qquad (\alpha_n)$$
$$\text{for all } j \quad r_1+\ldots+\hat{r}_j+\ldots+r_{p+1}\geq n \qquad (\beta_n)$$

Definition 4 The quantity $\kappa=2n^2-d_1-\ldots-d_{p+1}$ is called the *index of rigidity.* If condition (α_n) holds, then it takes the values 2, 0,-2, ... Call *rigid* the case $\kappa=2$ (i.e. for which condition (α_n) is an equality).

The rigid case has been studied in [Ka]. In the present paper we study the case $\kappa=0$. These two cases are of particular interest because they seem to contain all non-trivial examples when the DSP is not weakly solvable. (An example is called non-trivial if the JNFs J_j^n satisfy the conditions of Theorem 10 below.)

Definition 5 Denote by J_j^n the JNF of size n defined by the class c_j or C_j and by $\{J_j^n\}$ the $(p+1)$-tuple of these JNFs. For $n>1$ define the map $\Psi:\{J_j^n\}\mapsto\{J_j^{n_1}\}$ if the condition (β_n) holds and the condition

$$r_1+\ldots+r_{p+1}\geq 2n \qquad (\omega_n)$$

does not hold. Namely, set $n_1=(\sum_{j=1}^{p+1}r_j)-n$; hence, $n_1<n$. For each j the new JNF $J_j^{n_1}$ is defined after J_j^n by choosing an eigenvalue with the maximal possible number $n-r_j$ of Jordan blocks, by decreasing by 1 the sizes of the smallest $n-n_1$ of them and by deleting the Jordan blocks of size 0. One has $n-n_1\leq n-r_j$ because (β_n) holds. If there are several eigenvalues with maximal number of Jordan blocks, then we choose any of them.

Definition 6 A *multiplicity vector (MV)* is a vector whose components are non-negative integers whose sum is n. Notation: $\Lambda_j^n=(m_{1,j},\ldots,m_{i_j,j})$, $m_{1,j}\geq\ldots\geq m_{i_j,j}$, $m_{1,j}+\ldots+m_{i_j,j}=n$. The components have the meaning of the multiplicities of the eigenvalues of a matrix A_j or M_j (for the sake of convenience we admit components equal to 0). A *polymultiplicity vector (PMV)* is the $(p+1)$-tuple of MVs defined by the eigenvalues of the matrices A_j or M_j.

Remark 7 1) In the case of diagonalizable matrices A_j or M_j the JNF J_j^n is completely defined by the MV Λ_j^n and the construction Ψ results in decreasing the biggest component of Λ_j^n by $n-n_1$ to obtain $\Lambda_j^{n_1}$.

2) For a diagonal JNF defined by a MV Λ_j^n one has $r_j=n-m_{1,j}$ and $d_j=n^2-\sum_{\nu=1}^{i_j}m_{\nu,j}^2$.

3) If $\Lambda_j^n=(n)$ and if the matrix A_j or M_j is diagonalizable, then it is scalar.

2.2 Generic eigenvalues; non-genericity relations; the quantities l and ξ

We presume the necessary condition $\prod\det(C_j)=1$ (resp. $\sum\mathrm{Tr}(c_j)=0$) to hold. This means that the eigenvalues $\sigma_{k,j}$ (resp. $\lambda_{k,j}$) of the matrices from C_j (resp. c_j) repeated with their multiplicities, satisfy the condition

$$\prod_{k=1}^{n}\prod_{j=1}^{p+1}\sigma_{k,j}=1 \quad \text{resp.} \quad \sum_{k=1}^{n}\sum_{j=1}^{p+1}\lambda_{k,j}=0 \qquad (7)$$

An equality of the form

$$\prod_{j=1}^{p+1}\prod_{k\in\Phi_j}\sigma_{k,j}=1\ ,\quad \text{resp.}\quad \sum_{j=1}^{p+1}\sum_{k\in\Phi_j}\lambda_{k,j}=0\ ,$$

is called a *non-genericity relation*; the sets Φ_j contain one and the same number $< n$ of indices for all j. Eigenvalues satisfying none of these relations are called *generic*. Reducible $(p+1)$-tuples exist only for non-generic eigenvalues (a reducible $(p+1)$-tuple of matrices can be conjugated to a block upper-triangular form, its restriction to each diagonal block is such a $(p+1)$-tuple of smaller size, and, hence, the eigenvalues of each diagonal block satisfy condition (2) or (1) which is a non-genericity relation).

Remark 8 In the case of matrices A_j, if the greatest common divisor q of the multiplicities of all eigenvalues of all $p+1$ matrices is >1, then a non-genericity relation (γ_B) (called the *basic non-genericity relation*) results automatically from $\sum\mathrm{Tr}(c_j)=0$ when one decreases q times the multiplicities of all eigenvalues. In the case of matrices M_j the equality $\prod\sigma_{k,j}=1$ implies that if one divides by q the multiplicities of all eigenvalues, then their product would equal $\xi=\exp(2\pi ik/q)$, $0\leq k\leq q-1$, not necessarily 1. In this case a non-genericity relation holds exactly if ξ is a non-primitive root of unity of order q. Indeed, denote by l the greatest common divisor of q and k. Then the product of all eigenvalues with multiplicities divided by l equals 1 which is the *basic non-genericity relation* (γ_B) in the case of matrices M_j.

Definition 9 In the case when the basic non-genericity relation (γ_B) holds eigenvalues satisfying no non-genericity relation other than (γ_B) and its corollaries are called *relatively generic*.

The following theorem is the basic result from [Ko3], [Ko4] and [Ko5]:

Theorem 10 *Let $n>1$. The DSP is solvable for the conjugacy classes C_j or c_j (with generic eigenvalues, defining the JNFs J_j^n and satisfying conditions (α_n) and (β_n)) if and only if either $\{J_j^n\}$ satisfies condition (ω_n) or the construction $\Psi:\{J_j^n\}\mapsto\{J_j^{n_1}\}$ iterated as long as it is defined stops at a $(p+1)$-tuple $\{J_j^{n'}\}$ either with $n'=1$ or satisfying condition $(\omega_{n'})$.*

Proposition 11 *The construction Ψ preserves the index of rigidity.*

The proposition is proved in [Ko4].

Remark 12 1) The result of the theorem does not depend on the choice one makes in Ψ of an eigenvalue with maximal number of Jordan blocks (if such (a) choice(s) is (are) possible).

2) Proposition 11 implies that it suffices to check condition $(\alpha_{n'})$ for the $(p+1)$-tuple of JNFs $J_j^{n'}$ without checking (α_n) for the JNFs J_j^n. It does hold – if $n'=1$, then $(\alpha_{n'})$ is an equality (this is the *rigid* case, i.e. $\kappa=2$). If $n'>1$ and condition $(\omega_{n'})$ holds for the JNFs $J_j^{n'}$, then $(\alpha_{n'})$ holds and is a strict inequality, see [Ko3], Theorem 9. Thus a posteriori one knows that it is not necessary to check condition (α_n) in Theorem 10.

3 The basic result

3.1 The case $\kappa = 0$ for diagonalizable matrices

Lemma 13 *In the case $\kappa = 0$ a monodromy group with a trivial centralizer and with relatively generic eigenvalues is irreducible.*

The lemma is proved in [Ko5], see part 1) of Lemma 6 there. Making use of the lemma we shall not distinguish solvability from weak solvability of the DSP in the case $\kappa = 0$.

Theorem 14 *In the case of matrices M_j, for $\kappa = 0$, the conditions of Theorem 10 upon the JNFs J_j^n are necessary for the solvability of the DSP in the case $\kappa = 0$. If the conjugacy classes C_j defining the JNFs J_j^n satisfy condition (β_n) and do not satisfy condition (ω_n), then the solvability of the DSP for the conjugacy classes C_j implies the solvability of the DSP for the $(p+1)$-tuple of JNFs $J_j^{n_1} = \Psi(J_j^n)$ (see Subsection 2.1) for some relatively generic eigenvalues with the same value of ξ.*

The theorem is proved in Section 4. In order to announce the basic result we need to introduce some technical notions (see Subsections 3.2 and 3.3). Therefore we first announce the result for the case of diagonalizable matrices which does not need them.

Theorem 15 *1) If $\kappa = 0$, if the JNFs defined by the classes C_j are diagonal, if $q > 1$, if ξ is a non-primitive root of unity of order q and if the eigenvalues of the classes C_j are relatively generic, then the DSP is not weakly solvable for matrices M_j (hence, not solvable either).*

2) If $\kappa = 0$, if the JNFs defined by the classes c_j are diagonal, if $q > 1$ and if the eigenvalues of the classes c_j are relatively generic, then the DSP is not weakly solvable for matrices A_j (hence, not solvable either).

A plan of the proof of the theorem is given at the end of this subsection.

Remark 16 It is shown in [Ko5] that if the conditions of Theorem 10 upon the JNFs J_j^n are fulfilled and if ξ is a primitive root of unity of order q, then the DSP is weakly solvable for matrices M_j and $\kappa = 0$.

In the rigid case the construction Ψ stops at a $(p+1)$-tuple of one-dimensional JNFs, see Theorem 10 and Remark 12, part 2).

Lemma 17 *In the case when $\kappa = 0$ and the JNFs J_j^n are diagonal there are four possible $(p+1)$-tuples of JNFs at which Ψ stops. Their PMVs are:*

Case A)	$p = 3$	(d, d)	(d, d)	(d, d)	(d, d)
Case B)	$p = 2$	(d, d, d)	(d, d, d)	(d, d, d)	
Case C)	$p = 2$	(d, d, d, d)	(d, d, d, d)	$(2d, 2d)$	
Case D)	$p = 2$	(d, d, d, d, d, d)	$(2d, 2d, 2d)$	$(3d, 3d)$	

In all cases $d \in \mathbf{N}^*$; we assume that if when iterating Ψ there appears a MV of the form (n), then we delete it. In all four cases condition (ω_n) holds and is an equality.

The lemma follows from Lemma 3 from [Ko1] and from the notion of corresponding JNFs defined below in Subsection 3.3.

Plan of the proof of Theorem 15: We prove part 1) first. We show that in each of the four cases A) – D) from Lemma 17 (and when the conditions of 1) of the theorem are fulfilled) the DSP is not solvable; by Lemma 13 it is not weakly solvable either. This is done in Sections 5, 7, 6 and 8, one case per section. Section 5 is the longest and the most important of them because in the other three cases the proof is reduced to the one in Case A).

Theorem 14 and Lemma 17 imply that in all possible cases covered by Theorem 15 the DSP is not weakly solvable. Part 2) of the theorem is proved in Section 9 using part 1).

3.2 The basic technical tool

Definition 18 Call *basic technical tool* the way described below to deform analytically a $(p+1)$-tuple of matrices A_j satisfying (1) or of matrices M_j satisfying (2) with a *trivial centralizer*.

In the case of matrices A_j set $A_j = Q_j^{-1} G_j Q_j$, G_j being Jordan matrices. Look for matrices $\tilde{A}_j$ of the form $\tilde{A}_j = (I + \sum_{i=1}^{s} \varepsilon_i X_{j,i}(\varepsilon))^{-1} Q_j^{-1} (G_j + \sum_{i=1}^{s} \varepsilon_i V_{j,i}(\varepsilon)) Q_j (I + \sum_{i=1}^{s} \varepsilon_i X_{j,i}(\varepsilon))$ where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in (\mathbf{C}^s, 0)$ and $V_{j,i}(\varepsilon)$ are given matrices analytic in ε. One chooses $V_{j,i}$ such that $\mathrm{tr}(\sum_{j=1}^{p+1} \sum_{i=1}^{s} \varepsilon_i V_{j,i}(\varepsilon)) \equiv 0$ identically in ε. One often has $s = 1$ and $V_{j,1}$ are such that the eigenvalues of the $(p+1)$-tuple of matrices $\tilde{A}_j$ are generic for $\varepsilon \neq 0$. Often one has $V_{j,i} \equiv 0$ for all indices j but one, i.e. all matrices A_j but one remain within their conjugacy classes.

In the case of $(p+1)$-tuples of matrices M_j^1 with a trivial centralizer look for M_j of the form

$$M_j = (I + \sum_{i=1}^{s} \varepsilon_i X_{j,i}(\varepsilon))^{-1} (M_j^1 + \sum_{i=1}^{s} \varepsilon_i N_{j,i}(\varepsilon)) (I + \sum_{i=1}^{s} \varepsilon_i X_{j,i}(\varepsilon)) \qquad (8)$$

where the given matrices $N_{j,i}$ are analytic in $\varepsilon \in (\mathbf{C}^s, 0)$ and one looks for $X_{j,i}$ analytic in ε. Like in the case of matrices A_j one can set $M_j^1 = Q_j^{-1} G_j Q_j$, $N_{j,i} = Q_j^{-1} V_{j,i} Q_j$. For both cases the existence of the matrices $X_{j,i}$ analytic in ε is proved in [Ko4].

3.3 Correspondence between Jordan normal forms

Definition 19 For a given JNF $J^n = \{b_{i,k}\}$ define its *corresponding* diagonal JNF J'^n. A diagonal JNF is a partition of n defined by the multiplicities of the eigenvalues. For each k fixed the collection $\{b_{i,k}\}$ is a partition $\mathcal{P}_k$ of $\sum_{i \in I_k} b_{i,k}$. The diagonal JNF J'^n is the disjoint sum of the partitions dual to $\mathcal{P}_k$.

Example 20 Consider the JNF $J^{17} = \{\{6,4,3\}\{3,1\}\}$, i.e. with two eigenvalues, the first with three Jordan blocks of sizes 6,4,3 and the second with two blocks of sizes 3,1. The partition of 13 dual to (6,4,3) is (3,3,3,2,1,1), the one of 4 dual to (3,1) is (2,1,1). Hence, the diagonal JNF corresponding to J^{17} is defined by the MV (3,3,3,2,2,1,1,1,1) (in decreasing order of the multiplicities).

Proposition 21 *Consider a JNF J^n and its corresponding diagonal JNF J'^n defined by a MV $\Lambda = (m_1, \ldots, m_\nu)$, $m_1 \geq \ldots \geq m_\nu$. Choose an eigenvalue of J^n with maximal number $n - r(J^n)$ of Jordan blocks and decrease the sizes of the k' smallest of these blocks by 1, $k' \leq n - r(J^n)$ – this defines a new JNF $J^{n-k'}$. Set $\Lambda_* = (m_1 - k', m_2, \ldots, m_\nu)$. Then the MV Λ_* defines a diagonal JNF corresponding to $J^{n-k'}$.*

Corollary 22 *The $(p+1)$-tuples of JNFs J_j^n $J_j'^n$ where for each j J_j^n corresponds to $J_j'^n$ satisfy or not the conditions of Theorem 10 simultaneously.*

The propositions and corollary from this subsection are proved in [Ko4].

Proposition 23 *1) If the JNF J'^n corresponds to the JNF J^n, then $r(J^n) = r(J'^n)$ and $d(J^n) = d(J'^n)$.*

2) To each diagonal JNF there corresponds a unique JNF with a single eigenvalue.

Remark 24 Denote by G a Jordan matrix and by G' a diagonal matrix defined as follows: the diagonal entries of G' in the last but s positions of the Jordan blocks of G with given eigenvalue λ are equal among themselves and different from the ones in the last but m positions for $m \neq s$, $m, s \in \mathbf{N}^*$. Then the matrix $G+\varepsilon G'$, $0 \neq \varepsilon \in (\mathbf{C}, 0)$ is diagonalizable and its JNF is the diagonal JNF corresponding to $J(G)$ (the poof can be found in [Ko4]). Hence, if one applies the basic technical tool with $s = 1$ and G_j, $V_{j,1}$ playing the roles respectively of G, G', then one sees that the weak solvability of the DSP for matrices A_j or M_j with given JNFs J_j^n implies the one for diagonal JNFs corresponding to J_j^n and for nearby eigenvalues.

3.4 The result in the general case

Definition 25 We say that the conjugacy class C is *continuously deformed* into the class C' if either the classes C, C' are like the ones of the matrices G, $G + \varepsilon G'$ from Remark 24 or C' is just another conjugacy class defining the same JNF as C. We say that the $(p + 1)$-tuple of conjugacy classes C_j is continuously deformed into the $(p + 1)$-tuple of conjugacy classes C_j' if each class C_j is continuously deformed into the corresponding class C_j' and the eigenvalues of the first $(p + 1)$-tuple are homotopic to the ones of the second $(p + 1)$-tuple. Throughout the homotopy there holds condition (7) and the MVs remain the same.

Example 26 Consider the triple of conjugacy classes C_1, C_2, C_3 of size 12 each with a single eigenvalue λ_j and with Jordan blocks of equal size l_j: $(\lambda_1, \lambda_2, \lambda_3) = (i, 1, 1)$, $(l_1, l_2, l_3) = (2, 3, 6)$. For these eigenvalues one has $q = 12$, $\xi = i$ which is not a primitive root of unity of order 12. One has $l = 3$. The basic non-genericity relation (γ_B) is obtained by dividing the multiplicities of all eigenvalues by 3. The eigenvalues are relatively generic.

To the triple of JNFs defined by the conjugacy classes C_j there corresponds the triple of diagonal JNFs defined by the PMV $(6, 6)$, $(4, 4, 4)$, $(2, 2, 2, 2, 2, 2)$. For this PMV one has $q = 2$ and by continuous deformation of the conjugacy classes C_j into diagonal ones with the above PMV one obtains $\xi = -1$ which is a primitive root of unity of order 2. (Indeed, for the classes C_j the product of the eigenvalues repeated each with the half of its multiplicity equals -1 which remains unchanged throughout the continuous deformation.)

Definition 27 Denote by d the greatest common divisor of all quantities $\Sigma_{j,m}(\sigma)$ where $\Sigma_{j,m}(\sigma)$ is the number of Jordan blocks of size m of a given matrix M_j or A_j and with eigenvalue σ. It is true that d divides q and that q divides n.

Remark 28 The quantity q does not increase under continuous deformations like in the above example. If one deforms continuously the conjugacy classes so that the eigenvalues of C' be "as generic as possible" (i.e. satisfying only these non-genericity relations which are not destroyed by continuous deformations like the above ones), then one has $q = d$.

Theorem 29 *Suppose that*

1) the conjugacy classes of the matrices A_j or M_j verify the conditions of Theorem 10;

2) they are continuously deformed into a $(p+1)$-tuple of conjugacy classes defining diagonal JNFs with $q = d > 1$, with relatively generic eigenvalues and in the case of matrices M_j with ξ being a non-primitive root of unity of order q;

3) one has $\kappa = 0$.

Then for such conjugacy classes the DSP is not weakly solvable.

Proof: Suppose that there exists a $(p+1)$-tuple of matrices M_j with trivial centarlizer which satisfies conditions 1), 2) and 3). Applying the basic technical tool with $l = 1$ and G_j, $V_{j,1}$ like in Remark 24, one obtains the existence of a $(p+1)$-tuple of diagonalizable matrices M_j with a trivial centralizer, with relatively generic eigenvalues, with $\kappa = 0$ and with ξ being a non-primitive root of unity of order q which contradicts Theorem 15. $\square$

4 Proof of Theorem 14

4.1 The proof itself

Definition 30 A regular singular point of a linear system of ordinary differential equations is called *apparent* if its local monodromy is trivial.

Lemma 31 *Any monodromy group can be realized by a Fuchsian system on $\mathbf{C}P^1$ with at most one additional apparent singularity at a point a_{p+2} which can be chosen arbitrarily; for the eigenvalues $\lambda_{k,j}$ of the matrices-residua A_j, $j = 1, \ldots, p+1$ one has $\mathrm{Re}\lambda_{k,j} \in [0, 1)$; one has $J(A_j) = J(M_j)$ for $j = 1, \ldots, p+1$, M_j being the monodromy operators.*

The lemmas from this subsection except Lemmas 33 and 39 are proved in the subsequent ones (one proof per subsection). In what follows the points $a_1, \ldots, a_{p+2}$ are fixed.

Definition 32 A Fuchsian system belongs to the *class N* if it has poles at the points a_j the one at a_{p+2} being an apparent singularity, if its monodromy group is irreducible, and if at a_{p+2} the Laurent series expansion of the system looks like this:

$$\dot{X} = (A_{p+2}/(t - a_{p+2}) + B(t - a_{p+2}))X \tag{9}$$

where $A_{p+2} = \mathrm{diag}(\mu_1, \ldots, \mu_n)$, $\mu_j \in \mathbf{Z}$, $\mu_1 \geq \ldots \geq \mu_n$.

Denote by $\mathrm{ord}u$ the order of the zero at a_{p+2} of the germ of holomorphic function u. A class N Fuchsian system is called *normalized* if for $i < j$ one has $\mathrm{ord}B_{i,j} \geq \mu_i - \mu_j$.

Lemma 33 *If one has $A_{p+2} = \mathrm{diag}(\mu_1, \ldots, \mu_n)$, $\mu_j \in \mathbf{Z}$, $\mu_1 \geq \ldots \geq \mu_n$, and if one has for $i < j$ $\mathrm{ord}B_{i,j} \geq \mu_i - \mu_j$ for B defined by (9), then the singularity at a_{p+2} is apparent.*

Indeed, the following change of variables brings the system locally, at a_{p+2}, to a system without a pole at a_{p+2} (hence, the local monodromy at a_{p+2} is trivial):

$$X \mapsto (t - a_{p+2})^{\mathrm{diag}(\mu_1, \ldots, \mu_n)} X \tag{10}$$

Lemma 34 *For a normalized class N Fuchsian system one has $\mu_i - \mu_{i+1} \leq p$ for $i = 1, \ldots, n-1$.*

Definition 35 Set $\sigma = (\mu_1 + \ldots + \mu_n)/n$ (*mean value*) and $\delta = ((\mu_1 - \sigma)^2 + \ldots + (\mu_n - \sigma)^2)/n$ (*dispersion* of the numbers μ_i).

Lemma 36 *The monodromy group of a non-normalized class N Fuchsian system can be realized by a normalized class N Fuchsian system with the same conjugacy classes of the matrices* A_1, *…*, A_{p+1}, *with the same mean value and with a smaller dispersion of the numbers* μ_i.

Suppose that for $\kappa = 0$ and for given diagonal conjugacy classes with relatively generic eigenvalues and not satisfying condition (ω_n) there exists a monodromy group with a trivial centralizer (hence, irreducible by Lemma 13). Then for almost all relatively generic eigenvalues with the same value of ξ there exist irreducible monodromy groups with such JNFs. Indeed, applying the basic technical tool, one can deform the given monodromy group into one with any nearby relatively generic eigenvalues and the same JNFs of the matrices M_j. Moreover, the deformation can be chosen such that the new matrices M_j will be diagonalizable and defining the JNFs corresponding to the initial ones.

The set $\mathcal{M}$ of such monodromy groups is constructible and such is its projection $\mathcal{V}$ on the set of eigenvalues $\mathcal{W}$, i.e. $\mathcal{V}$ is an everywhere dense constructible subset of $\mathcal{W}$.

Lemmas 31, 34 and 36 imply that for given conjugacy classes C_j of M_1, …, M_{p+1} there exist finitely many sets Γ_i of eigenvalues $\mu_k = \lambda_{k,p+2}$ such that the monodromy group can be realized by a normalized class N Fuchsian system with such eigenvalues of A_{p+2}; for $j \leq p+1$ the eigenvalues $\lambda_{k,j}$ are uniquely defined by the classes C_j, see Lemma 31.

Consider $gl(n, \mathbf{C})^{p+1}$ as the space of $(p+2)$-tuples of matrices A_j whose sum is 0. Denote by $\mathcal{G}_i$ its subsets such that A_{p+2} is diagonal, with eigenvalues $\mu_k \in \Gamma_i$, and for $i < j$ there holds the condition $\mathrm{ord} B_{i,j} \geq \mu_i - \mu_j$ for B defined by (9) (recall that the poles a_j are fixed). Hence, the sets $\mathcal{G}_i$ are constructible.

A point from $\mathcal{G}_i$ defines a Fuchsian system (S). Fix a base point a different from the points a_j and define the monodromy operators of the system with initial data $X|_{t=a} = I$. The map which maps the matrices-residua A_1, …, A_{p+2} into the $(p+1)$-tuple of monodromy operators of system (S) is a map $\chi_i : \mathcal{G}_i \to \mathcal{M}$.

For each point from $\mathcal{M}$ there exists at least one i such that the point has a preimage in $\mathcal{G}_i$ under χ_i. This means that there exists a point from $\mathcal{M}$ such that some neighbourhood of his is covered by $\chi_i(\mathcal{G}_i)$ for some i; we set $i = 1$. Indeed, the constructible set $\mathcal{M}$ cannot be locally covered by a finite number of analytic sets of lower dimension. This and the irreducibility of $\mathcal{M}$ implies that the set $\chi_1(\mathcal{G}_1)$ is dense in $\mathcal{M}$.

Lemma 37 *Suppose that*

A) the matrices-residua A_1, *…*, A_{p+1} *of a normalized class N Fuchsian system are diagonalizable, with generic eigenvalues;*

B) their $(p+1)$*-tuple is irreducible;*

C) none of these matrices has eigenvalues differing by a non-zero integer and each of them has a single integer eigenvalue λ_j *whose multiplicity is a (the) greatest one (hence, each monodromy operator* M_j *has an eigenvalue* $\sigma_j = 1$*);*

D) all non-genericity relations satisfied by the eigenvalues of the monodromy operators M_j *result from two relations, the first of which is the basic one* (γ_B) *the second being*

$$\sigma_1 \ldots \sigma_{p+1} = 1 \qquad (\gamma_0)$$

E) one has $\lambda_j > 0$ *and* $\lambda_1 + \ldots + \lambda_{p+1} > (n^2 + n)\mu$ *with* $\mu = \max(|\mu_1|, |\mu_n|)$.

F) the monodromy group can be analytically deformed into an irreducible one for nearby relatively generic eigenvalues and with the same JNFs of the matrices M_j.

G) Condition (ω_n) does not hold for the matrices M_j.

Then the monodromy group of the Fuchsian system is with trivial centralizer.

The projection $\mathcal{P}_1$ of the set $\mathcal{G}_1$ on the space $\mathbf{C}^s$ of eigenvalues $\lambda_{k,j}$ (s depends on their multiplicities) is a constructible set. If $\mathcal{P}_1$ does not contain a point satisfying conditions C), D) and E) of the lemma, then $\mathrm{codim}_{\mathbf{C}^s}\mathcal{P}_1 > 0$, hence, $\chi_1(\mathcal{G}_1)$ cannot be dense in $\mathcal{M}$.

Lemma 38 *The monodromy group of system (4) with eigenvalues defined as in Lemma 37 can be conjugated to the form* $\begin{pmatrix} \Phi & * \\ 0 & I \end{pmatrix}$ *where* Φ *is* $n_1 \times n_1$*.*

The subrepresentation Φ can be reducible. The following lemma is proved in [Ko4].

Lemma 39 *The centralizer* $\mathcal{Z}(\Phi)$ *of the subrepresentation* Φ *is trivial.*

Thus the existence of an irreducible representation of rank n for which condition (ω_n) does not hold implies the existence of the representation Φ of rank n_1 and with trivial centralizer. The JNFs defined by the matrices from Φ are obtained from the initial $(p+1)$ JNFs by applying the map Ψ. One can deform the eigenvalues of Φ so that they become relatively generic. For such eigenvalues the deformed representation Φ is irreducible, see Lemma 13. If Φ satisfies condition (ω_{n_1}), then we are done. If not, then we continue iterating Ψ. In the end we stop at a representation of rank n' satisfying condition $(\omega_{n'})$. It is impossible to obtain a representation of rank 1 because its index of rigidity is 2, see Proposition 11.

The eigenvalues of the representation Φ define the same value of ξ as the ones of the initial representation. Indeed, the eigenvalues from the initial one which are not in Φ equal 1. $\square$

4.2 Proof of Lemma 31

It is shown in [P] that any monodromy group can be realized by a regular system on $\mathbf{C}P^1$ which is Fuchsian at all poles but one. So one can add a $(p+2)$-nd monodromy operator equal to I to the initial operators M_j assuming that the system realizing this monodromy group has not $p+1$ but $p+2$ poles. Applying the result from [P] (reproved in [ArIl], p. 131) one obtains a regular system (S) with the given monodromy group which is Fuchsian at $a_1, \ldots, a_{p+1}$ and which has a regular apparent singularity at a_{p+2}. The point $a_{p+2} \neq a_j$, $j \leq p+1$, is chosen arbitrarily and the JNFs of the matrices A_j are the same as the ones of the corresponding monodromy operators M_j for $j = 1, \ldots, p+1$. Moreover, $\mathrm{Re}\lambda_{k,j} \in [0, 1)$.

Remark 40 In [P] an attempt is made to prove that every monodromy group can be realized by a Fuchsian system on $\mathbf{C}P^1$ (without apparent singularities). This is one of the versions of the Riemann-Hilbert problem and the answer to it is negative, see [Bo1]. We are referring above to the correct part of the attempt from [P] to prove the Riemann-Hilbert problem. See [ArIl] pp. 130 – 135 as well.

Make the singularity at a_{p+2} Fuchsian. Fix a matrix solution to system (4) with $\det X \not\equiv 0$. Its regularity and the triviality of the monodromy at a_{p+2} imply that it is meromorphic at a_{p+2}.

Lemma 41 *(A. Souvage) A meromorphic mapping from* $\mathbf{C}^n$ *to* $\mathbf{C}^n$ *with a pole at* a_{p+2} *and nondegenerate for* $t \neq a_{p+2}$ *can be represented in the form* $PH(t-a_{p+2})^D$ *where* D *is a diagonal matrix with integer entries,* H *is holomorphic and holomorphically invertible at* a_{p+2} *and the entries of the matrix* P *are polynomials in* $1/(t-a_{p+2})$*,* $\det P \equiv$*const*$\neq 0$.

Perform in system (S) the change $X \mapsto P^{-1}X$. This change leaves the system Fuchsian at $a_1, \ldots, a_{p+1}$ and regular at a_{p+2} without introducing new singular points. At a_{p+2} the new system is Fuchsian. Indeed, the matrix $(t-a_{p+2})^D$ is a solution to the system (Fuchsian at a_{p+2}) $\dot{X} = (D/(t-a_{p+2}))X$. The change of variables $X \mapsto HX$ leaves the latter system Fuchsian at a_{p+2} (the system becomes $\dot{X} = (-H^{-1}\dot{H} + H^{-1}(D/(t-a_{p+2}))H)X$). $\square$

4.3 Proof of Lemma 34

1^0. The matrix B defined by equation (9) admits the Taylor series expansion $B = B_0 + (t-a_{p+2})B_1 + (t-a_{p+2})^2B_2 + \ldots$. A direct computation shows that $B_\nu = -\sum_{j=1}^{p+1} A_j/(a_j - a_{p+2})^\nu$. Suppose that for some i_0 ($1 \leq i_0 \leq n-1$) one has $\mu_{i_0} - \mu_{i_0+1} \geq p+1$. Then for $i \leq i_0$, $k \geq i_0+1$ one has $\mu_i - \mu_k \geq p+1$.

2^0. Hence, all matrix entries $A_{j;i,k}$ with $j \leq p+1$ and i,k like in 1^0 must be 0. Indeed, for each such i,k fixed the system of linear equations $B_{\nu;i,k} = 0$, $\nu = 1, \ldots, p+1$ with unknown variables the entries $A_{j;i,k}$ implies $A_{j;i,k} = 0$ because it is of rank $p+1$ (its determinant is the Vandermonde one $W(1/(a_1-a_{p+2}), \ldots, 1/(a_{p+1}-a_{p+2}))$ and for $j_1 \neq j_2$ one has $a_{j_1} \neq a_{j_2}$).

This means that the matrices-residua $A_1, \ldots, A_{p+1}$ are block lower-triangular, with diagonal blocks of sizes i_0 and $n-i_0$. Hence, so are the monodromy operators, i.e. the monodromy group is reducible and the system is not from the class N. $\square$

4.4 Proof of Lemma 36

1^0. Recall that the matrix B was defined by equation (9). Assume for simplicity that $a_{p+2} = 0$. For $i < j$ find an entry $B_{i,j}$ with smallest value of $m := -\text{ord}B_{i,j} - \mu_j + \mu_i$. Hence, $m > 0$. If there are several possible choices, then we choose among them one with minimal value of $j - i$. Set $B_{i,j} = bt^g + o(|t|^g)$, $b \neq 0$ (hence, $g = \text{ord}B_{i,j}$).

2^0. Consider the change of variables $X \mapsto WX$ with $W = I + (\mu_j - \mu_i + g)E_{j,i}/bt^m$. It is holomorphic for $t \neq 0$, with $\det W \equiv 1$, hence, it preserves the conjugacy classes of the residua $A_1, \ldots, A_{p+1}$ the system remaining Fuchsian there. At a_{p+2} the new residuum is lower-triangular, with diagonal entries equal to $\mu_1, \ldots, \mu_{i-1}, \mu_j + g, \mu_{i+1}, \ldots, \mu_{j-1}, \mu_i - g, \mu_{j+1}, \ldots, \mu_n$. The singularity at a_{p+2}, in general, is no longer Fuchsian, but the order of the pole at a_{p+2} is $\leq m$; equality is possible only in position (j,i). This follows from rule (6) (the reader is invited to check the claim).

Except on the diagonal poles of order > 1 at 0 can appear only in the entries $(j,1)$, $(j,2)$, $\ldots$, (j,i), $(j+1,i)$, $(j+2,i)$, $\ldots$, (n,i), see the choice of $B_{i,j}$ in 1^0.

3^0. One deletes the polar terms below the diagonal by a change $X \mapsto VX$, $V = I + V'$ where each entry $V'_{k,\nu}$ of V' is a suitably chosen polynomial $p_{k,\nu}$ of $1/t$, the non-zero entries being in the positions cited at the end of 2^0. The degree of the polynomial $p_{k,\nu}$ is equal to the order of the pole in position (k,ν) which has to disappear. We leave for the reader the proof that such a choice of the polynomials $p_{k,\nu}$ is really possible.

4^0. As a result of the changes from 2^0 and 3^0 the system remains Fuchsian at a_j for $j \leq p+1$ and the conjugacy classes of its residua do not change because the matrix V is holomorphic for $t \neq 0$ and $\det V \equiv 1$. The system remains Fuchsian at 0 as well and the eigenvalues of A_{p+2} change as follows: $\mu_i \mapsto \mu_i - g$, $\mu_j \mapsto \mu_j + g$, the rest of the eigenvalues remain the same. (One

should rearrange after this the eigenvalues μ_i in decreasing order by conjugating with a constant permutation matrix.) One checks directly that as a result of the change of the eigenvalues μ_i the mean value σ remains the same whereas δ decreases. $\square$

4.5 Proof of Lemma 37

1^0. Suppose that the centralizer $\mathcal{Z}$ is nontrivial. Hence, it contains either a diagonalizable matrix D with exactly two different eigenvalues or a nilpotent matrix $N \neq 0$ such that $N^2 = 0$.

2^0. Suppose that $D = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix} \in \mathcal{Z}$ with diagonal blocks of sizes l' and $n - l'$ and with $\alpha \neq \beta$. Then the matrices M_j are block-diagonal with the same sizes of the diagonal blocks and the monodromy group is a direct sum. This follows from $[M_j, D] = 0$. Denote the two diagonal blocks of M_j by S_j and T_j (S_j is $l' \times l'$).

Hence, there are two subspaces of the solution space ($\mathcal{X}_1$ and $\mathcal{X}_2$) which are invariant for the monodromy group and whose direct sum is the solution space. Denote by C_j', C_j'' the conjugacy classes of the matrices S_j and T_j.

3^0. Use a result from [Bo1] (see Lemma 3.6 there):

Lemma 42 *The sum of the eigenvalues $\lambda_{k,j}$ of the matrices-residua A_j corresponding to an invariant subspace of the monodromy group is a non-positive integer.*

Remark 43 1) Condition C) and Remark 1 imply that the equality $\exp(2\pi i \lambda_{k,j}) = \sigma_{k,j}$ defines (for $j \leq p+1$ fixed) a bijection between the eigenvalues $\sigma_{k,j}$ and the eigenvalues $\lambda_{k,j}$ modulo permutation of equal eigenvalues. For $j = p+2$ this is false (recall that $\lambda_{k,p+2} = \mu_k \in \mathbf{Z}$, $\sigma_{1,p+2} = \ldots = \sigma_{n,p+2} = 1$).

2) When defining the sets of eigenvalues $\lambda_{k,j}$ corresponding to the subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ it is true only for $j \leq p+1$ but not for $j = p+2$ that these sets are complementary to one another, i.e. one and the same eigenvalue $\lambda_{k,p+2} = \mu_k$ might appear in both sums while another one might appear in none of them.

Indeed, present the eigenvalues $\lambda_{k,j}$ in the form $\varphi_{k,j} + \rho_{k,j}$ with $\varphi_{k,j} \in \mathbf{Z}$, $\mathrm{Re}\rho_{k,j} \in [0,1)$ (this presentation is unique). The numbers $\varphi_{k,j}$ have the meaning of valuations on the solution subspace on which the monodromy operator M_j acts with a single eigenvalue $\exp(2\pi i \rho_{k,j})$, see the details in [Bo1] (Definition 2.3 etc.).

At a_{p+2} one has $\rho_{k,p+2} = 0$, $\varphi_{k,p+2} = \mu_k$. Thus if a vector-column solution $\tilde{X}' \in \mathcal{X}_1$ of system (4) has an expansion at a_{p+2} into a Laurent series $v_1(t-a_{p+2})^{\mu_{i_1}} + v_2(t-a_{p+2})^{\mu_{i_2}} + o((t-a_{p+2})^{\mu_{i_2}})$, with $\mu_{i_1} < \mu_{i_2}$ and $0 \neq v_i \in \mathbf{C}^n$, then it is μ_{i_1} that participates in the sum of eigenvalues $\lambda_{k,j}$ corresponding to $\mathcal{X}_1$ because this is the valuation of $\tilde{X}'$ at a_{p+2}.

If a solution $\tilde{X}'' \in \mathcal{X}_2$ equals $cv_1(t - a_{p+2})^{\mu_{i_1}} + dv_2(t - a_{p+2})^{\mu_{i_2}} + o((t - a_{p+2})^{\mu_{i_2}})$, $c, d \in \mathbf{C}^*$, $c \neq d$, then it is again μ_{i_1} that participates in the sum corresponding to $\mathcal{X}_2$. The number μ_{i_2} is a valuation of the solution $c\tilde{X}' - \tilde{X}''$ which might be neither in $\mathcal{X}_1$ nor in $\mathcal{X}_2$, therefore μ_{i_2} might appear in neither of the two sums. For $j \leq p+1$ there is no such ambiguity due to condition C), i.e. to each eigenvalue of the monodromy operator M_j there corresponds a single valuation on the corresponding solution subspace.

4^0. *Lemma 42 and conditions D) and E) imply that if the monodromy group is a direct sum, then equal eigenvalues of the matrices S_j and T_j have proportional multiplicities.*

Indeed, denote by Ξ, Θ the sets of eigenvalues $\sigma_{k,j}$, $j \leq p+1$ participating respectively in (γ_B), (γ_0) and by Ξ', Θ' the sums of their respective eigenvalues $\lambda_{k,j}$. Hence, the sums of

eigenvalues of the matrices $A_1, \ldots, A_{p+2}$ relative to the solution subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are both of the form $\phi_i := a_i\Xi' + b_i\Theta' + \Delta_i$, $a_i \in \mathbf{N}$, $b_i \in \mathbf{Z}$, $b_1 + b_2 = 0$ where Δ_1 (resp. Δ_2) is the sum of some l' (resp. $n - l'$) eigenvalues $\lambda_{k,p+2} = \mu_k$ (see Remark 43); hence, $|\Delta_i| \leq n\mu$.

One has $a_i \leq n$ (evident), and $|\Xi'| < n\mu$ (because the sum of all eigenvalues $\lambda_{k,j}$ (which is 0) is of the form $g\Xi' + \sum_{k=1}^{n}\mu_k$ with $g \in \mathbf{N}$, $1 < g < n$; hence, $|\Xi'| \leq n\mu/g < n\mu$).

If $b_1 > 0$, then $\phi_1 \geq b_1(n^2+n)\mu - a_1|\Xi'| - |\Delta_1| > (n^2+n)\mu - n^2\mu - n\mu > 0$. This contradicts Lemma 42. Hence, $b_1 \leq 0$. In the same way $b_2 \leq 0$. Hence, $b_1 = b_2 = 0$. This means that equal eigenvalues of the blocks S_j and T_j have proportional multiplicities.

5^0. The monodromy group of a Fuchsian system satisfying the condition $b_1 = b_2 = 0$, see 4^0, cannot be analytically deformed into an irreducible one for nearby relatively generic eigenvalues and with the same Jordan normal forms of the matrices M_j; this contradicts condition F).

Indeed, suppose that there exists such a deformation analytic in $\varepsilon \in (\mathbf{C}, 0)$ (i.e. for almost all values of $\varepsilon \neq 0$ the $(p+1)$-tuple is irreducible). For the $(p+1)$-tuple before the deformation the multiplicities of the equal eigenvalues $\sigma_{k,j}$ of the two diagonal blocks S_j and T_j are proportional for all j. This means that for all j one has $d(C_j') = (l'^2/n^2)d(C_j)$, $d(C_j'') = ((n-l')^2/n^2)d(C_j)$. Indeed, if a diagonal JNF is defined by the PMV $(m_1, \ldots, m_s)$, then a conjugacy class defining such a JNF is of dimension $n^2 - \sum_{i=1}^{s}(m_i)^2$. Hence, $d(C_1'') + \ldots + d(C_{p+1}'') = 2(n-l')^2$, $d(C_1') + \ldots + d(C_{p+1}') = 2l'^2$ (this follows from the proportional multiplicities) and for the representations $\mathcal{M}'$, $\mathcal{M}''$ defined by the matrices S_j, T_j one has

$$\mathrm{Ext}^1(\mathcal{M}', \mathcal{M}'') = \mathrm{Ext}^1(\mathcal{M}'', \mathcal{M}') = 0 \tag{11}$$

6^0. When one deforms analytically a $(p+1)$-tuple into a nearby one (see the basic technical tool) one can express the deformation as a superposition of two deformations – of a change of the eigenvalues (see the matrices $N_{j,i}(\varepsilon)$ in (8)) and of a conjugation (see the matrices $X_{j,i}(\varepsilon)$ there). One can choose the matrices $N_{j,i}$ to be polynomials of the matrices M_j, i.e. block-diagonal, with diagonal blocks of sizes l' and $n - l'$. Hence, the two non-diagonal blocks of the matrices change (in first approximation w.r.t. ε) only as a result of the conjugation.

Condition (11) shows that up to conjugacy the $(p+1)$-tuple remains block-diagonal in first approximation w.r.t. ε. Hence, one can conjugate it by a matrix analytic in ε to make the non-diagonal blocks zero in first approximation w.r.t. ε. In the same way one shows that the $(p+1)$-tuple is block-diagonal up to conjugacy of any order w.r.t. ε. The deformation being analytic, the $(p+1)$-tuple is block-diagonal up to conjugacy for ε small enough and non-zero – a contradiction.

7^0. If there exists $N \in \mathcal{Z}$ like in 1^0, then one can conjugate the matrix N and the matrices M_j to the form $N = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $M_j = \begin{pmatrix} P_j & U_j & V_j \\ 0 & Q_j & W_j \\ 0 & 0 & P_j \end{pmatrix}$ where the middle row and column of blocks might be absent. If they are absent, then the monodromy group is a direct sum. Indeed, for the conjugacy classes C_j' of the matrices P_j one has $d(C_j') = d(C_j)/4$, hence, $d(C_1') + \ldots + d(C_{p+1}') = n^2/2$, see 5^0. One has (11) with $\mathcal{M}' = \mathcal{M}''$ being the representation defined by the matrices P_j. Hence, the monodromy group is indeed a direct sum.

8^0. Suppose that the middle row and column of blocks are present. Lemma 42 and conditions D) and E) imply that the multiplicities of the eigenvalues of the matrices M_j for the diagonal blocks P_j and Q_j are proportional. Indeed, the blocks $S_j = P_j$ (the upper P_j) and $T_j = \begin{pmatrix} P_j & U_j \\ 0 & Q_j \end{pmatrix}$ define invariant subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ of the monodromy group. Like in the case when $D \in \mathcal{Z}$, see 1^0, and using the same notation one shows that equal eigenvalues of the

matrices S_j and T_j are of proportional multiplicities. This implies that there holds (11), hence, the monodromy group is a direct sum of the groups defined by the blocks S_j and T_j, i.e. after a simultaneous conjugation of the matrices M_j one has $V_j = W_j = 0$. Hence, there exists $D \in \mathcal{Z}$ like in 1^0 which possibility is already rejected.

4.6 Proof of Lemma 38

1^0. The monodromy group can be conjugated to a block upper-triangular form. The diagonal blocks define either irreducible or one-dimensional representations. The eigenvalues of each diagonal block 1×1 satisfy the non-genericity relation (γ_0) from Lemma 37.

2^0. **The lowest diagonal block is of size 1.**

Indeed, set $M_j = \begin{pmatrix} Q_j & * \\ 0 & L_j \end{pmatrix}$ where L_j is the restriction of M_j to the lowest diagonal block (say, of size h). Denote by Ξ, Θ the sets of eigenvalues $\sigma_{k,j}$, $j \leq p+1$ participating respectively in (γ_B), (γ_0) and by Ξ', Θ' the sums of their respective eigenvalues $\lambda_{k,j}$. Hence, the set of eigenvalues of the blocks L_1, …, L_{p+1} is of the form $a\Xi + b\Theta$, $a \in \mathbf{N}$, $b \in \mathbf{Z}$.

If $a > 0$, $b \geq 0$, then condition (β_h) is not fulfilled by the blocks L_j (this condition is necessary because these blocks define an irreducible monodromy group of $h \times h$-matrices). Indeed, for $b = 0$ it is not fulfilled because it is not fulfilled by the matrices M_j and the multiplicities of equal eigenvalues of M_j and L_j are proportional. When increasing h, i.e. when increasing $b \in \mathbf{Z}$ while keeping a fixed it is only the biggest multiplicity that increases and it is of an eigenvalue equal to 1. Hence, the sum of the quantities r_j computed for the matrices L_j remains the same while their size h increases.

On the other hand, one cannot have $b < 0$ because in this case the sum of the eigenvalues $\lambda_{k,j}$ corresponding to the invariant solution subspace on which the monodromy group acts with the blocks Q_j would be positive which contradicts Lemma 42. Indeed, the sum of these eigenvalues equals $\phi := c\Xi' - b\Theta' + \Delta$ where Δ is the sum of some $n - h$ eigenvalues of the matrix A_{p+2}. We prove that $\phi > 0$ like we prove that $\phi_1 > 0$ in 4^0 of the proof of Lemma 37.

3^0. Denote by Π the left upper $(n-1) \times (n-1)$-block. Conjugate it to make all non-zero rows of the restriction of the $(p+1)$-tuple $\tilde{M}$ of matrices $M_j - I$ to Π linearly independent. After the conjugation some of the rows of the restriction of $\tilde{M}$ to Π might be 0. In this case conjugate the matrices M_j by one and the same permutation matrix which places the zero rows of $M_j - I$ in the last (say, m) positions (recall that the last row of $M_j - I$ is 0, see 2^0, so $m \geq 1$). Notice that if the restriction to Π of a row of $M_j - I$ is zero, then its last (i.e. n-th) position is 0 as well, otherwise M_j is not diagonalizable.

4^0. There remains to show that $m \geq n - n_1$. One has $M_j = \begin{pmatrix} G_j & R_j \\ 0 & I \end{pmatrix}$, $I \in GL(m, \mathbf{C})$. Denote by $\tilde{G}$ the representation defined by the matrices G_j. We regard the columns of the $(p+1)$-tuple of matrices R_j as elements of the space $\mathcal{F}(\tilde{G})$ (or just $\mathcal{F}$ for short) defined as follows. Set $U^* = (U_1, \ldots, U_{p+1})$. Set $\mathcal{D} = \{U^* | U_j = (G_j - I)V_j, V_j \in \mathbf{C}^m, \sum_{j=1}^{p+1} G_1 \ldots G_{j-1} U_j = 0\}$, $\mathcal{E} = \{U^* | U_j = (G_j - I)V, V \in \mathbf{C}^m\}$, $\mathcal{F} = \mathcal{D}/\mathcal{E}$.

Remark 44 If $R_j = (G_j - I)V$ with $V \in \mathbf{C}^m$ or with $V \in M_{m,n-m}$, then there holds

$$\sum_{j=1}^{p+1} G_1 \ldots G_{j-1} R_j = 0 \qquad (12)$$

One has $\mathcal{E} \subset \mathcal{D}$. Equality (12) with $V \in M_{m,n-m}$ is condition (2) restricted to the block R.

5^0. Each column of the $(p+1)$-tuple of matrices R_j belongs to the linear space $\mathcal{D}$. **The latter is of dimension $\theta = r_1 + \ldots + r_{p+1} - (n-m)$.**

Indeed, the image of the linear operator $\tau_j : (.) \mapsto (G_j - I)(.)$ acting on $\mathbf{C}^{n-m}$ is of dimension r_j (every column of R_j belongs to the image of this operator, otherwise M_j will not be diagonalizable). The $n-m$ linear equations resulting from (12) with $R_j = U_j = (G_j - I)V$, $V \in \mathbf{C}^m$ are linearly independent.

Indeed, if they are not, then the images of all linear operators τ_j must be contained in a proper subspace of $\mathbf{C}^{n-m}$ (say, the one defined by the first $n-m-1$ vectors of its canonical basis). This means that all entries of the last rows of the matrices $G_j - I$ are 0. The matrices M_j being diagonalizable, this implies that the entire $(n-m)$-th rows of $M_j - I$ are 0. This contradicts the condition the first $n-m$ rows of the restriction to Π of the $(p+1)$-tuple of matrices $M_j - I$ to be linearly independent, see 3^0.

6^0. **The space $\mathcal{F}$ is of codimension $n-m$ in $\mathcal{D}$, i.e. of dimension $\theta - 2(n-m)$.**

Indeed, each vector-column V belongs to $\mathbf{C}^{n-m}$ and the intersection $\mathcal{I}$ of the kernels of the operators τ_j is $\{0\}$, otherwise the matrices M_j would have a non-trivial common centralizer – if $\mathcal{I} \neq \{0\}$, then after a change of the basis of $\mathbf{C}^{n-m}$ one can assume that a non-zero vector from $\mathcal{I}$ equals ${}^t(1, 0, \ldots, 0)$. Hence, the matrices G_j are of the form $\left(\begin{array}{cc} 1 & * \\ 0 & G_j^* \end{array} \right)$, $G_j^* \in GL(n-m-1, \mathbf{C})$, and one checks directly that $[M_j, E_{1,n}] = 0$ for $E_{1,n} = \{\delta_{i-1,n-j}\}$.

7^0. The columns of the $(p+1)$-tuple of matrices R_j (regarded as elements of $\mathcal{F}$) must be linearly independent, otherwise the monodromy group can be conjugated by a matrix $\left(\begin{array}{cc} I & * \\ 0 & P \end{array} \right)$, $P \in GL(m, \mathbf{C})$, to a block-diagonal form in which the right lower blocks of M_j are equal to 1, the monodromy group is a direct sum and, hence, its centralizer is non-trivial – a contradiction. This means that $\dim \mathcal{F} = \theta - 2(n-m) = r_1 + \ldots + r_{p+1} - 2(n-m) \geq m$ which is equivalent to $m \geq n - n_1$; recall that $n_1 = r_1 + \ldots + r_{p+1} - n$. In the case of equality (and only in it) the columns of the $(p+1)$-tuple of matrices R_j are a basis of the space $\mathcal{F}$. $\Box$

5 Case A)

In this section we prove

Theorem 45 *The DSP is not solvable (hence, not weakly solvable, see Lemma 13) for quadruples of diagonalizable matrices M_j each with MV equal to $(n/2, n/2)$ where $n \geq 4$ is even, the eigenvalues are relatively generic and ξ is a non-primitive root of unity of order $n/2$.*

Remark 46 *In case A) for relatively generic eigenvalues there exist only block-diagonal quadruples of matrices M_j with diagonal blocks $(n/l) \times (n/l)$. Their existence follows from [Ko5], Theorem 3. The non-existence of others follows from Theorem 45.*

The proof of the theorem consists of three steps. We assume that irreducible quadruples as described in the theorem exist. The first step is a preliminary deformation and conjugation of the quadruple which brings in some technical simplifications, the quadruple remaining irreducible and satisfying the conditions of the theorem, see the next subsection. At the second step we discuss the possible eigenvalues of the matrix $M_1 M_2$ after the first step, see Subsection 5.2. At the third step we prove that the new quadruple must be reducible, see Subsection 5.3.

5.1 Preliminary conjugation and deformation

Set $S = M_1M_2 = (M_4)^{-1}(M_3)^{-1}$. Denote by g_j, h_j the eigenvalues of M_j.

Lemma 47 *The triple M_1, M_2, S^{-1} admits a conjugation to a block upper-triangular form with diagonal blocks of sizes only 1 or 2. The restriction of the triple to each diagonal block of size 2 is irreducible.*

Indeed, suppose that the triple is in block upper-triangular form, its restrictions to each diagonal block being irreducible (in particular, the triple can be irreducible, i.e. with a single diagonal block). The restriction of M_j to each diagonal block (say, of size k) is diagonalizable and has eigenvalues g_j and h_j, of multiplicities l^0 and $k - l^0$. Hence, the conjugacy class of the restriction of M_j to the block is of dimension $2l^0(k - l^0) \leq k^2/2$.

An irreducible triple with such blocks of M_1 and M_2 of size $k > 1$ can exist only for $k = 2$, in all other cases condition (α_k) does not hold. Indeed, the conjugacy class of the restriction of S to the diagonal block is of dimension $\leq k^2 - k$. Hence, the sum of the three dimensions is $\leq k^2/2 + k^2/2 + k^2 - k = 2k^2 - k$ which is $< 2k^2 - 2$ if $k > 2$. □

Give a more detailed description of the diagonal blocks of the triple M_1, M_2, S^{-1} after the conjugation (in the form of lemmas; Lemmas 48, 51 and 52 are to be checked directly).

Lemma 48 *1) There are four possible representations defined by diagonal blocks of size 1 of the triple; we list them by indicating the couples of diagonal entries respectively of M_1 and M_2:*

$$P \; g_1, g_2 \quad ; \quad Q \; h_1, h_2 \quad ; \quad R \; g_1, h_2 \quad ; \quad U \; h_1, g_2 \quad .$$

2) Denote by V and W any two of these couples. For a given V there exists a unique W (denoted by $-V$) such that the corresponding diagonal entries of both M_1 and M_2 are different. One has $P = -Q$ and $R = -U$.

3)One has dim $Ext^1(V, W) = 1$ if and only if $V = -W$. In the other cases one has dim $Ext^1(V, W) = 0$.

Lemma 49 *There are equally many diagonal blocks of type V as there are of type $-V$.*

Indeed, consider first the case when there are no blocks of size 2. Denote by p', q', r' and u' the number of blocks P, Q, R and U. The multiplicities of the eigenvalues imply that $p' + r' = p' + u' = q' + u' = q' + r' = n/2$. Hence, $r' = u'$ and $p' = q'$.

If there are blocks of size 2, then each of them contains once each of the eigenvalues g_1, g_2, h_1, h_2 and the proof is finished in the same way as in the particular case considered above. □

Lemma 50 *In an irreducible representation defined by a 2×2-block the eigenvalues of S can equal any couple (λ, μ) (with $\lambda\mu = g_1h_1g_2h_2$) which is different from (g_1g_2, h_1h_2) and (g_1h_2, g_2h_1).*

Indeed, one can show (the easy computation is omitted) that if the eigenvalues of S equal g_1g_2, h_1h_2 or g_1h_2, g_2h_1, then the triple is triangular up to conjugacy. On the other hand, if one fixes M_1 =diag(g_1, h_1) and varies M_2 within its conjugacy class, one can obtain any trace of the product M_1M_2. The determinant of the product being fixed, this means that M_1M_2 can belong to any non-scalar conjugacy class the product of whose eigenvalues equals $g_1h_1g_2h_2$. (The choice of the eigenvalues excludes the possibility S to be scalar.) □

Lemma 51 *The semi-direct sums defined by two diagonal blocks of size 1 are up to conjugacy of one of the types:* $(M_1, M_2) = \left(\begin{pmatrix} g_1 & r \\ s & h_1 \end{pmatrix}, \begin{pmatrix} g_2 & r' \\ s' & h_2 \end{pmatrix} \right)$ or $\left(\begin{pmatrix} g_1 & u \\ m & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & u' \\ m' & g_2 \end{pmatrix} \right)$ *with either $r = r' = 0$ or $s = s' = 0$ but not both (resp. with either $u = u' = 0$ or $m = m' = 0$ but not both). Such semi-direct sums exist only for couples $(V, -V)$, see 2) and 3) from Lemma 48. The centralizers of these semi-direct sums are trivial.*

Denote by Φ, Ψ respectively an irreducible representation of rank 2 defined by a diagonal block of the triple M_1, M_2, S^{-1} and a representation which is either irreducible and non-equivalent to Φ or one-dimensional (i.e. of type P, Q, R or U, see Lemma 48) or a semi-direct sum of two one-dimensional ones $(V, -V)$, see Lemmas 48 and 51.

Lemma 52 *One has dim* $Ext^1(\Phi, \Psi) =$*dim* $Ext^1(\Psi, \Phi) = 0$.

Definition 53 We say that the triple M_1, M_2, S^{-1} or M_3, M_4, S is in a *special form* if it is block-diagonal, each diagonal block B_μ being itself block upper-triangular, its diagonal blocks being of equal size which is either 1 or 2. In the case of size 2 all diagonal blocks of each block B_μ define equivalent representations. In the case of size 1 the block B_μ is of size 2 and defines a semi-direct sum, see Lemma 51. Thus a triple in special form is block upper-triangular with diagonal blocks of size 2 defining either irreducible representations or semi-direct sums like in Lemma 51.

Lemma 54 *One can deform the matrices M_j within their conjugacy classes (without changing the matrix S) so that after the deformation each of the triples M_1, M_2, S^{-1} and M_3, M_4, S after a suitable conjugation is in special form. The two conjugations are, in general, different.*

The lemma is proved in Subsection 5.4.

5.2 The possible eigenvalues of the matrix S

The eigenvalues of the matrix S (even when they are distinct) must satisfy certain equalities – for every diagonal block of size 2 (irreducible or not) of the triple M_1, M_2, S^{-1} (resp. M_3, M_4, S) the eigenvalues λ, μ of S must satisfy the condition $g_1h_1g_2h_2\lambda^{-1}\mu^{-1} = 1$ (resp. $g_3h_3g_4h_4\lambda\mu = 1$).

In what follows we denote the eigenvalues of S by s_i. Let the triple M_1, M_2, S^{-1} (resp. M_3, M_4, S) be in special form. For each eigenvalue s_i denote by $t(s_i)$ (resp. by $u(s_i)$) the eigenvalue of S in the same diagonal 2×2-block of the triple with s_i. Note that $t(t(s_i)) = s_i = u(u(s_i))$. One has $t(s_i) = u(s_i)$ if and only if $\xi = 1$ (and this holds for all $i = 1, \ldots, n$).

Set $i_1 = 1$. For the eigenvalue $s_1 = s_{i_1}$ find $s_{i_2} \stackrel{\text{def}}{=} t(s_{i_1})$, then find $s_{i_3} \stackrel{\text{def}}{=} u(s_{i_2})$, then $s_{i_4} \stackrel{\text{def}}{=} t(s_{i_3})$, then again $s_{i_5} \stackrel{\text{def}}{=} u(s_{i_4})$ etc. Thus one has $s_{i_{\nu+1}} = t(s_{i_\nu})$ for ν odd (hence, $t(s_{i_{\nu+1}}) = t(t(s_{i_\nu})) = s_{i_\nu}$) and $s_{i_{\nu+1}} = u(s_{i_\nu})$ for ν even (hence, $u(s_{i_{\nu+1}}) = u(u(s_{i_\nu})) = s_{i_\nu}$.

Denote by m the least value of α for which one has $i_\alpha = 1$. It is clear that $m - 1$ is even.

Lemma 55 *For ν odd one has $s_{i_{\nu+1}} = \xi s_{i_{\nu-1}}$, for ν even one has $s_{i_{\nu+1}} = \xi^{-1}s_{i_{\nu-1}}$.*

Indeed, there holds $g_1h_1g_2h_2s_{i_\nu}^{-1}(t(s_{i_\nu}))^{-1} = g_3h_3g_4h_4s_{i_\nu}u(s_{i_\nu}) = 1$ and $\prod_{j=1}^4 g_jh_j = \xi$. Hence, $\xi^{-1}t(s_{i_\nu}) = u(s_{i_\nu})$. For ν odd this yields $\xi^{-1}s_{i_{\nu+1}} = \xi^{-1}t(s_{i_\nu}) = u(s_{i_\nu}) = u(u(s_{i_{\nu-1}})) = s_{i_{\nu-1}}$, for ν even in the same way it gives $\xi^{-1}s_{i_{\nu-1}} = s_{i_{\nu+1}}$. $\square$

Lemma 56 *One has $m-1<n/2$ and $m-1$ divides $n/2$.*

Proof: Recall that $\xi = \exp(2k\pi i/(n/2)) = \exp(4k\pi i/n)$ (see Subsection 2.2). If $k=0$, i.e. $\xi = 1$, then $s_3 = s_1$, i.e. $m-1=1$, and the statement holds.

Let $k \neq 0$. Then $s_{1+(m-1)} = (\xi)^{-m+1}s_1 = s_1$ (Lemma 55). Hence, $(\xi)^{-m+1} = 1$, i.e. $4k(m-1) = 2nl$ (l is defined in Subsection 2.2), i.e. $k(m-1) = (n/2)l$. The minimality of m (hence, of $m-1$ as well) implies that $m-1$ and l are relatively prime, i.e. $m-1$ divides $n/2$. The non-primitivity of ξ implies $k>1$. Hence, $m-1<n/2$. $\square$

Remark 57 Lemma 56 implies that the set of eigenvalues of S can be partitioned into $n/(2m-2)$ sets $\mathcal{N}_1$, …, $\mathcal{N}_{n/(2m-2)}$ each consisting of $(2m-2)$ eigenvalues (denoted again by s_i) with the properties $s_{2k+2} = \xi s_{2k}$, $s_{2k+1} = \xi^{-1}s_{2k-1}$, $s_{2k-1}s_{2k} = g_1h_1g_2h_2$ and $s_{2k}^{-1}s_{2k+1}^{-1} = g_3h_3g_4h_4$. If some of the sets $\mathcal{N}_i$ are identical, then we define their *multiplicities* in a natural way. Two non-identical sets $\mathcal{N}_i$ have no eigenvalue in common. In what follows we change the indexation – equal (different) indices indicate identical (different) sets $\mathcal{N}_i$.

5.3 End of the proof of Theorem 45

Case 1) *The matrix S has at least two different sets $\mathcal{N}_i$.*

Then the upper-triangular form of the triple M_1, M_2, S is in addition block-diagonal, the restrictions of the matrix S to two different diagonal blocks having no eigenvalue in common. Indeed, it suffices to rearrange the blocks B_μ from the special form putting first all the blocks B_μ with eigenvalues of S from $\mathcal{N}_1$ (repeated with its multiplicity – this defines the diagonal block R_1), then all blocks with eigenvalues of S from $\mathcal{N}_2$ (this defines the diagonal block R_2) etc. The size of the block R_i equals l_i times the number of eigenvalues from $\mathcal{N}_i$, $l_i \in \mathbf{N}^*$.

The triple M_3, M_4, S admits a conjugation to the same block-diagonal form. Hence, if the triple M_1, M_2, S^{-1} is block-diagonal (with diagonal blocks R_i), to give the same form of the triple M_3, M_4, S one has to use as conjugation matrix one commuting with S, hence, a block-diagonal one with diagonal blocks of the sizes of the blocks R_i. Hence, both triples are simultaneously block-diagonal, i.e. the quadruple M_1, M_2, M_3, M_4 is block -diagonal, i. e. reducible.

Case 2) *There is a single set $\mathcal{N}_1$ repeated $n/(2m-2)$ times.* In this case one can deform the matrices M_j, $j=1,2$, so that the matrix S have at least two different sets $\mathcal{N}_i$ of eigenvalues.

Definition 58 We say that a matrix is in s-block-diagonal (resp. in s-block upper-triangular) form if it is block-diagonal (resp. block upper-triangular) with diagonal blocks all of size s.

Set $\mu = n/(2m-2)$. Conjugate the triple M_1, M_2, S to a $(2m-2)$-block upper-triangular form where the diagonal blocks of the matrix S are with eigenvalues from $\mathcal{N}_1$:

$$M_j = \begin{pmatrix} M_j' & H_{j;1,2} & \ldots & H_{j;1,\mu} \\ 0 & M_j' & \ldots & H_{j;2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_j' \end{pmatrix} \quad , \quad j=1,2 \quad , \quad S = \begin{pmatrix} T & Q_{1,2} & \ldots & Q_{1,\mu} \\ 0 & T & \ldots & Q_{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & T \end{pmatrix} \quad .$$

We assume that the blocks M_j' and T are 2-block-diagonal.

Deform analytically the left upper blocks of size $2m-2$ of the matrices M_1, M_2 and S so that they remain 2-block-diagonal and the eigenvalues of S change to new ones, forming again a set of $2m-2$ eigenvalues like in Remark 57 but different from $\mathcal{N}_1$. To this end one can keep the

matrix M_1 the same and vary the left upper block of the matrix M_2; see Lemma 50. This block will become $M_2' + \varepsilon U$, $\varepsilon \in (\mathbf{C}, 0)$, $U \in gl(2m-2, \mathbf{C})$, and the one of S will equal $M_1(M_2' + \varepsilon U)$. The other blocks of M_1, M_2 and S do not change.

One can deform in a similar way the triple of matrices M_3^{-1}, M_4^{-1} S (requiring the deformation of S to be the same in both triples). For $\varepsilon \neq 0$ small enough the quadruple of matrices remains irreducible. However, there are already two different sets $\mathcal{N}_i$ of eigenvalues of S, so we are in Case 1) and the quadruple is block-diagonal. Hence, the initial quadruple is also reducible.

5.4 Proof of Lemma 54

Notation 59 Assume that the triple M_1, M_2, S satisfies the conclusion of Lemma 47. Block-decompose each matrix from $gl(n, \mathbf{C})$ the sizes of the diagonal blocks being the same as the ones of the triple M_1, M_2, S. Denote the block of this decomposition in the i-th row and k-th column of blocks by $([i, k])$. By (i, k) we denote the matrix entry in the i-th row and k-th column.

1^0. *Up to conjugacy the triple M_1, M_2, S is block-diagonal, with two diagonal blocks (T and Y) which are block upper-triangular, their diagonal blocks being respectively of size 1 and 2, the latter defining irreducible representations.*

Indeed, whenever a block $([i, i+1])$ of the triple M_1, M_2, S is of size 1×2 or 2×1, it can be made equal to 0 by a simultaneous conjugation of the triple with a matrix of the form $I + R$ where only the block $([i, i+1])$ of R is non-zero. This follows from Lemma 52. After this in the same way one annihilates all blocks $([i, i+2])$ of size 1×2 or 2×1, then all blocks $([i, i+3])$ of these sizes etc. Then one rearranges the diagonal blocks putting the ones of size 1 first and the ones of size 2 next. This gives the claimed form.

2^0. *The block Y after conjugation becomes block-diagonal, its diagonal blocks B_μ being block upper-triangular, their diagonal blocks being of size 2. The diagonal blocks of one and the same (resp. of different) diagonal blocks B_μ define equivalent (resp. non-equivalent) representations.*

This is proved by analogy with 1^0, making use of Lemma 52.

3^0. Denote by $V_1, \ldots, V_n$ the diagonal blocks of T.

One can conjugate the triple M_1, M_2, S by an upper-triangular matrix so that after the conjugation only these blocks $([i, j])$, $i < j$, remain possibly non-zero for which $V_i = -V_j$.

This is proved like 1^0 and 2^0, making use of 2) and 3) of Lemma 48.

4^0. *After a conjugation and deformation the block T of the triple M_1, M_2, S becomes block-diagonal, with upper-triangular diagonal blocks of size 2 defining semi-direct sums, see Lemma 51.*

The proof of this statement occupies $4^0 - 5^0$. It completes the proof of the lemma.

A conjugation of the triple M_1, M_2, S with a permutation matrix places the set of blocks P and Q first and the set of blocks R and U last on the diagonal; the triple remains block upper-triangular, in addition it is block-diagonal, the sizes of the diagonal blocks equal respectively $\sharp P + \sharp Q$ and $\sharp R + \sharp U$ (one of these sizes can be 0).

It suffices to consider the case when only, say, blocks P and Q are present, in the general case the reasoning is the same. Observe first that the blocks P and Q can be situated on the diagonal in any possible order.

The eigenvalues of the restrictions of S to the blocks P and Q being different, one can conjugate the triple with an upper-triangular matrix to make S diagonal. Moreover, all blocks $([i, j])$, $i < j$, with $V_i = V_j$ are 0, otherwise at least one of the matrices M_1, M_2 will not be diagonal.

5^0. Consider first the case when the triple after this conjugation becomes diagonal. Rearrange the blocks in alternating order – P, Q, P, Q, ... Make non-zero the entries $(1,2)$, $(3,4)$, $(5,6)$ etc. of the matrices M_j without changing the matrix S. With the notation from Lemma 51 this amounts to choosing $s=s'=0$, $r\neq 0$, $r'=-rh_2g_1^{-1}$ (look at the first couple (M_1,M_2) from the lemma). This gives the necessary block-diagonal form of the block T. The representations P and Q being non-equivalent, the centralizers of the diagonal blocks are trivial.

Suppose now that the triple is not diagonalizable and that $V_1=P$ (the case $V_1=Q$ is considered by analogy). Denote by $i_1<\ldots<i_h$ the indices i for which $V_i=Q$. Denote by m the *smallest* i_ν for which at least one of the entries (k,m) of M_1 and M_2 is non-zero, $k<m$; by 3^0, k is not among the indices i_ν. Denote the *greatest* such value of k by k_0. Hence, all entries (i,k_0) $(i<k_0)$ and (k_0,μ) $(\mu<m)$ of M_1 and M_2 are 0, otherwise these matrices will not be diagonalizable.

One can annihilate all entries (k',m) of M_j where $k'<k_0$ by consecutively conjugating the triple M_1,M_2,S by matrices of the form $I+gE_{k',k_0}$. Note that the values of k' are not among the indices i_ν. In a similar way one annihilates all entries (k_0,k'') of M_j with $k''>m$ by consecutive conjugations with matrices of the form $I+gE_{m,k''}$.

Hence, it is possible to conjugate the triple by a permutation matrix putting the k_0-th and m-th rows and columns first and preserving its upper-triangular form; in addition, the triple will be block-diagonal with first diagonal block of size 2 (which is upper-triangular non-diagonal and with trivial centralizer). After this one continues in the same way with the lower block. In the end the block T will become upper-triangular and block-diagonal, with diagonal blocks of size 2 each of which is triangular non-diagonal with trivial centralizer.

6 Case C)

Lemma 60 *If $\kappa=0$ and if the DSP is solvable for a $(p+1)$-tuple of conjugacy classes C_j with relatively generic eigenvalues defining the diagonal JNFs J_j^n, then the DSP is solvable for any $(p+1)$-tuple of JNFs $J_j'^n$ and for any relatively generic eigenvalues with the same value of ξ where for each j the JNFs J_j^n and $J_j'^n$ correspond to one another or are the same.*

The lemma is proved at the end of the subsection.

Assume that there exist irreducible triples of diagonalizable matrices M_j such that $M_1M_2M_3=I$, the PMV of the eigenvalues of the matrices being equal to (d,d,d,d), (d,d,d,d), $(2d,2d)$. Denote by $\sigma_{k,j}$ the eigenvalues of M_j where $k=1,2,3,4$ if $j=1$ or 2 and $k=1,2$ if $j=3$.

One can choose the eigenvalues of M_1 and M_2 such that $\sigma_{1,j}=-\sigma_{2,j}$ and $\sigma_{3,j}=-\sigma_{4,j}$, $j=1,2$, see Lemma 60. Hence, the MVs of the eigenvalues of the matrices $(M_1)^2$ and $(M_2)^2$ are of the form $(2d,2d)$. Set $A=M_1M_2=(M_3)^{-1}$, $B=M_2M_1$. The matrix B is conjugate to $(M_3)^{-1}$ (because $B=M_2(M_3)^{-1}(M_2)^{-1}$). One has $AB=M_1(M_2)^2M_1$, hence, $AB=(M_1)^2(M_1)^{-1}(M_2)^2M_1$. Set

$$L_1=A=M_1M_2 \quad , \quad L_2=B=M_2M_1 \quad , \quad L_3=(M_1)^{-1}(M_2)^{-2}M_1 \quad , \quad L_4=(M_1)^{-2} \ .$$

One has $L_1L_2L_3L_4=I$. The matrices L_j are diagonalizable, their MVs equal $(2d,2d)$ and by Case A) they define a block-diagonal algebra $\mathcal{C}$ with $2k$ blocks $2s\times 2s$. Hence, dim$\mathcal{C}\leq 8ks^2$.

The algebra $\mathcal{C}$ contains also the matrices $(L_j)^{-1}$. Hence, it contains the matrices $(M_1)^2=(L_4)^{-1}$, $M_1M_2=L_1$, $M_2M_1=L_2$ and $(M_2)^2=(M_2M_1)(L_3)^{-1}(M_2M_1)^{-1}$.

Every matrix from the algebra $\mathcal{D}$ generated by M_1 and M_2 is of the form $K+M_1L+M_2S$ with $K,L,S\in\mathcal{C}$. Hence, dim$\mathcal{D}\leq 3$dim$\mathcal{C}<n^2=$dim$gl(n,\mathbf{C})$. By the Burnside theorem, the matrix algebra $\mathcal{D}$ is reducible.

Proof of Lemma 60: 1^0. Suppose that the DSP is not solvable for the JNFs J'^n_j and for some relatively generic but not generic eigenvalues. Prove that then it is not solvable for the JNFs J^n_j and for any such eigenvalues. Note first that the JNFs J^n_j and J'^n_j satisfy the conditions of Theorem 10, see Corollary 22.

2^0. An irreducible $(p+1)$-tuple $\mathcal{H}$ of matrices M_j with JNFs J^n_j can be realized by a Fuchsian system with diagonalizable matrices-residua A_j such that $J(A_j) = J(M_j)$ for $j \leq p+1$ and with an additional apparent singularity, see Subsection 4.1 with the definition of the sets $\mathcal{G}_i$, the maps χ_i and $\mathcal{M}$. One can choose i such that $\chi_i(\mathcal{G}_i)$ is dense in $\mathcal{M}$.

3^0. Vary the eigenvalues of the matrices A_j within the set $\mathcal{G}_i$ without changing their JNFs. For suitable eigenvalues (in general, with integer differences between some of them; see 5^0) one obtains as monodromy group $\mathcal{H}'$ of the Fuchsian system one in which either $J(M_j) = J'^n_j$ or $J(M_j)$ is *subordinate* to J'^n_j, i.e. the multiplicities of the eigenvalues are the same and for each eigenvalue λ and for each $s \in \mathbf{N}$ $\mathrm{rk}(M_j - \lambda I)^s$ is the same or smaller than should be, see the details in [Ko4]. One can assume that the eigenvalues of the matrices M_j are relatively generic. Such a monodromy group cannot be irreducible (otherwise one could deform it using the basic technical tool into a nearby one with the same eigenvalues and with $J(M_j) = J'^n_j$ for all j; such irreducible monodromy groups do not exist by assumption).

4^0. The monodromy group $\mathcal{H}'$ can be analytically deformed into the monodromy group $\mathcal{H}$ because both are obtained from the Fuchsian system for different eigenvalues of the matrices-residua. However, $\mathcal{H}'$ cannot be analytically deformed into a nearby irreducible monodromy group with JNFs as in $\mathcal{H}$.

Indeed, if for all j one has $J(M_j) = J'^n_j$ in $\mathcal{H}'$, then the monodromy group $\mathcal{H}'$ must be block-diagonal with diagonal blocks of equal size and for the representations Φ_1, Φ_2 defined by two diagonal blocks one has $\mathrm{Ext}^1(\Phi_1, \Phi_2) \leq 0$ with equality if and only if Φ_1, Φ_2 are not equivalent. The last inequality holds also if for some j $J(M_j)$ is subordinate to J'^n_j. After this one applies the reasoning from $5^0 - 8^0$ of the proof of Lemma 37.

5^0. It is explained in [Ko4] how to choose the eigenvalues from 3^0 to obtain the monodromy group $\mathcal{H}'$ with $J(M_j)$ equal or subordinate to J'^n_j. Their possible choice is not unique – if one adds to equal eigenvalues of the matrices A_j equal integers the sum of all added integers (taking into account the multiplicities) being 0, then one obtains a new possible such set of eigenvalues; different eigenvalues of a given matrix A_j must remain such and if two eigenvalues of a given matrix A_j differ by a non-zero integer, then the order of their real parts must be preserved.

From all these a priori possible choices there is at least one which is really possible, i.e. for which there exists such a point from $\mathcal{G}_i$. Indeed, $\mathcal{G}_i$ is constructible and its projection on the set of eigenvalues $\mathcal{W}$ must be dense in $\mathcal{W}$, see Subsection 4.1. □

7 Case B)

Definition 61 A *special triple* is an irreducible triple of matrices M_j such that $M_1M_2M_3 = I$, $M_1 - I$ and $M_2 - I$ being conjugate to nilpotent Jordan matrices consisting each of $n/3$ Jordan blocks of size 3, M_3 being diagonalizable, with three eigenvalues each of multiplicity $n/3$. The eigenvalues are presumed to be relatively generic but not generic.

In the present subsection we prove that special triples do not exist. By Lemma 60, there exist no irreducible triples from Case B).

Lemma 62 *Suppose that there exist special triples. Then there exist special triples satisfying the conditions*

i) $Im(M_j - I) \cap \mathrm{Ker}(M_{2-j} - I) = \{0\}$, $j = 1, 2$

ii) $\mathbf{C}^n = \mathrm{Ker}(M_1 - I) \oplus \mathrm{Ker}(M_2 - I) \oplus (\mathrm{Im}(M_1 - I) \cap \mathrm{Im}(M_2 - I))$.

Corollary 63 *If there exist special triples, then there exist special triples in which the matrices* $M_1 - I$, $M_2 - I$ *are of the form* $M_1 - I = \left(\begin{array}{ccc} 0 & 0 & 0 \\ P & 0 & 0 \\ Q & R & 0 \end{array} \right)$, $M_2 - I = \left(\begin{array}{ccc} 0 & I & V \\ 0 & 0 & I \\ 0 & 0 & 0 \end{array} \right)$ *in which all blocks are* $(n/3) \times (n/3)$, *the matrices* P *and* R *being non-degenerate.*

The lemmas and the corollary from this section are proved at its end. Let the matrices M_j be like in Corollary 63. Consider the matrices

$$N_1 = \left(\begin{array}{ccc} I & 0 & V-I \\ 0 & I & I \\ 0 & 0 & I \end{array} \right) ~,~ N_2 = \left(\begin{array}{ccc} I & 0 & 0 \\ P & I & 0 \\ Q & 0 & I \end{array} \right) ~,~ G = \left(\begin{array}{ccc} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) ~,~ H = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{array} \right) .$$

Hence, each of the matrices $N_1 - I$, $N_2 - I$, $G + H$, G and H is nilpotent and conjugate to a Jordan matrix consisting of $n/3$ blocks of size 2 and of $n/3$ blocks of size 1. One has (to be checked directly)

$$N_2(I + H) = M_1 ~~,~~ \text{i.e. } M_1 - I = (N_2 - I)(I + H) + (G + H) - G \qquad (13)$$

$$(I + G)N_1 = M_2 ~~,~~ \text{i.e. } M_2 - I = (I + G)(N_1 - I) + G \qquad (14)$$

$$GH = G^2 = H^2 = HG = 0 ~~,~~ (N_1 - I)G = 0 ~~,~~ (N_2 - I)H = 0 \qquad (15)$$

Hence, $N_2(I + G + H)N_1 = M_1M_2$. Denote by $\mathcal{A}$ the matrix algebra generated by the matrices $N_1 - I$, $G + H$ and $N_2 - I$, by $\mathcal{B}$ the one generated by M_1 and M_2.

Lemma 64 *The matrix algebra* $\mathcal{A}$ *is reducible and* $dim\mathcal{A} \leq n^2/2$.

One has $\mathcal{B} = \mathcal{A} + G\mathcal{A} + \mathcal{A}G + G\mathcal{A}G + G\mathcal{A}G\mathcal{A} + \mathcal{A}G\mathcal{A}G + \ldots$ $(*)$. Indeed, every product of the matrices $M_1 - I$ and $M_2 - I$ (in any order and quantity) is representable as a linear combination of such products of the matrices $N_1 - I$, $N_2 - I$, $G + H$ and G, see (13), (14) and (15).

On the other hand, one has $\mathcal{A}G \subset \mathcal{A}$. Indeed, denote by Y a product of the matrices $N_1 - I$, $N_2 - I$, $G + H$ (in any quantity and order). If its right most factor is $N_1 - I$ or $G + H$, then by (15) one has $YG = 0$. If it is $N_2 - I$, then $YG = Y(G + H) - YH = Y(G + H) \in \mathcal{A}$.

This together with $(*)$ implies that $\mathcal{B} = \mathcal{A} + G\mathcal{A}$ $(**)$. Suppose that the couple of matrices M_1, M_2 is irreducible. Then by the Burnside theorem the algebra $\mathcal{B}$ equals $gl(n, \mathbf{C})$, i.e. $\dim\mathcal{B} = n^2$. The restriction of each matrix from $\mathcal{B}$ to the last $2n/3$ rows is the restriction to them of a matrix from $\mathcal{A}$, see $(**)$. This means that $\dim\mathcal{A} \geq 2n^2/3$ which contradicts Lemma 64. Hence, special triples do not exist.

Proof of Lemma 62: 1^0. Recall that the three conjugacy classes C_j of the matrices M_j belong to $SL(n, \mathbf{C})$. Denote by $\mathcal{U}$ the variety of irreducible representations (i.e. triples (M_1, M_2, M_3) defined up to conjugacy) where $M_j \in C_j \subset SL(n, \mathbf{C})$, $M_1M_2M_3 = I$.

Find $\dim\mathcal{U}$. One has to consider the cartesian product $C_1 \times C_2 \subset (SL(n, \mathbf{C}) \times SL(n, \mathbf{C}))$. The algebraic variety $\mathcal{V} \subset (SL(n, \mathbf{C}))^2$ of irreducible couples of matrices M_1, M_2 such that $M_1 \in C_1$, $M_2 \in C_2$ and $(M_1M_2)^{-1} \in C_3$ is the projection in $C_1 \times C_2$ of the intersection of the two varieties in $C_1 \times C_2 \times SL(n, \mathbf{C})$: the cartesian product $C_1 \times C_2 \times C_3$ and the graph of the mapping $(C_1 \times C_2) \ni (M_1, M_2) \mapsto M_3 = M_2^{-1}M_1^{-1} \in SL(n, \mathbf{C})$. This intersection

is transversal which implies the smoothness of the variety $\mathcal{V}$ (this can be proved by analogy with 1) of Theorem 2.2 from [Ko2]). Thus $\dim \mathcal{V} = (\sum_{j=1}^{2} \dim C_j) - [(n^2-1) - \dim C_3]$ (here $(n^2-1) - \dim C_3 = \mathrm{codim}_{SL(n,\mathbf{C})} C_3$). Hence, $\dim\mathcal{V} = \dim C_1 + \dim C_2 + \dim C_3 - n^2 + 1$.

2^0. In order to obtain $\dim\mathcal{U}$ from $\dim\mathcal{V}$ one has to factor out the possibility to conjugate the triple (M_1, M_2, M_3) with matrices from $SL(n, \mathbf{C})$. No non-scalar such matrix commutes with all the matrices (M_1, M_2, M_3) due to the irreducibility of the triple and to Schur's lemma. Thus $\dim \mathcal{U} = \dim \mathcal{V} - \dim SL(n, \mathbf{C}) = \sum_{j=1}^{3} \dim C_j - 2n^2 + 2 = 2$.

3^0. The subvariety $\mathcal{U}' \subset \mathcal{U}$ on which one has dim $(\mathrm{Ker}(M_j - I) \cap \mathrm{Im}(M_{2-j} - I)) > 0$ for $j = 1,\ 2$ is of positive codimension in $\mathcal{U}$. Indeed, its dimension is computed like the one of $\mathcal{U}$, by replacing the cartesian product $C_1 \times C_2$ by its subvariety on which one has dim $(\mathrm{Ker}(M_j - I) \cap \mathrm{Im}(M_{2-j} - I)) > 0$ for $j = 1,\ 2$. This subvariety is of positive codimension. Hence, the condition dim $(\mathrm{Ker}(M_j - I) \cap \mathrm{Im}(M_{2-j} - I)) > 0$ for $j = 1,\ 2$ cannot hold for all points from $\mathcal{U}$.

Condition *ii)* follows from condition *i)*. □

Proof of Corollary 63: 1^0. One has $\dim \mathrm{Ker}(M_1 - I) = \dim \mathrm{Ker}(M_2 - I) = n/3$. Condition *ii)* of Lemma 62 implies that dim $(\mathrm{Im}(M_1 - I) \cap \mathrm{Im}(M_2 - I)) = n/3$; recall that $\mathrm{Ker}(M_j - I) \subset \mathrm{Im}(M_j - I)$, $j = 1, 2$. Choose a basis of $\mathbf{C}^n$ such that the first $n/3$ vectors are a basis of $\mathrm{Ker}(M_2 - I)$, the next $n/3$ vectors are a basis of $\mathrm{Im}(M_1 - I) \cap \mathrm{Im}(M_2 - I)$ and the last $n/3$ vectors are a basis of $\mathrm{Ker}(M_1 - I)$. Hence, in this basis the matrices of $M_1 - I$, $M_2 - I$ look like this: $M_1 - I = \begin{pmatrix} 0 & 0 & 0 \\ P' & T & 0 \\ Q' & R' & 0 \end{pmatrix}$, $M_2 - I = \begin{pmatrix} 0 & W & V' \\ 0 & U & Y \\ 0 & 0 & 0 \end{pmatrix}$ (all blocks are $(n/3) \times (n/3)$).

2^0. One has $(M_2 - I)^3 = \begin{pmatrix} 0 & WU^2 & WUY \\ 0 & U^3 & U^2Y \\ 0 & 0 & 0 \end{pmatrix} = 0$. The rank of the matrix $\begin{pmatrix} W \\ U \end{pmatrix}$ equals $n/3$ because $\mathrm{rk}(M_2 - I) = 2n/3$. Therefore the equalities $\begin{pmatrix} WU^2 \\ U^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} WUY \\ U^2Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ imply respectively $U^2 = 0$ and $UY = 0$. It follows from $\mathrm{rk}(M_2 - I) = 2n/3$ that $\mathrm{rk}(U\ Y) = n/3$. Hence, the equality $(U^2\ UY) = (0\ 0)$ implies $U = 0$.

3^0. In the same way one proves that $T = 0$. A simultaneous conjugation of $M_1 - I$ and $M_2 - I$ with the matrix $\begin{pmatrix} WY & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & I \end{pmatrix}$ brings them to the desired form. Note that $\det W \neq 0 \neq \det Y$ and $\det P' \neq 0 \neq \det R'$ due to $\mathrm{rk}(M_1 - I) = \mathrm{rk}(M_2 - I) = 2n/3$. Hence, $\det P \neq 0 \neq \det R$. □

Proof of Lemma 64: Recall that one has $N_2(I + G + H)N_1 = M_1M_2$ and that the matrix M_1M_2 is diagonalizable with three eigenvalues each of multiplicity $n/3$. Hence, the quadruple of matrices N_2, $I + G + H$, N_1 and $(M_1M_2)^{-1}$ (their product is I) is reducible – if the map Ψ is applied to the quadruple, then one obtains a quadruple of conjugacy classes of size $2n/3$ the first three of which are each with a single eigenvalue and with $n/3$ Jordan blocks of size 2 and the fourth of which is diagonalizable, with two eigenvalues each of multiplicity $n/3$. One can apply the basic technical tool to such a quadruple and deform it into one with relatively generic but not generic eigenvalues and in which all four matrices are diagonalizable and have two eigenvalues of multiplicity $n/3$. This is a quadruple from Case A) (recall that the value of ξ is preserved), hence, block-diagonal up to conjugacy with diagonal blocks of one and the same size (Remark 46).

Hence, there exist only block-diagonal up to conjugacy quadruples of matrices N_2, $I + G + H$, N_1 and $(M_1M_2)^{-1}$ and all their diagonal blocks are of the same size. The dimension of such a

matrix algebra is $\leq n^2/2$ with equality if and only if there two diagonal blocks.

8 Case D)

Set $s = n/l$ (l was defined in Subsection 2.2). Hence, $n = 6ks$, $k > 1$ and the MVs of M_1, M_2, M_3 equal respectively (sk, sk, sk, sk, sk, sk), $(2sk, 2sk, 2sk)$, $(3sk, 3sk)$. Case D) can be reduced to Case B) like this: if the DSP is solvable in case D), then using Lemma 60 one can choose the eigenvalues of M_3 to be ± 1, i.e. $(M_3)^2 = I$, and the ones of M_1 to form three couples of opposite eigenvalues; hence, the MV of $(M_1)^2$ is $(2sk, 2sk, 2sk)$ and one has $(M_1)^{-2} = M_2(M_3M_2M_3)$.

Hence, the three matrices $(M_1)^2$, M_2 and $M_3M_2M_3 = (M_3)^{-1}M_2M_3$ are from Case B). By assumption, they define a block diagonal matrix algebra $\mathcal{A}$ with $2k$ diagonal blocks $3s \times 3s$ (Remark 46). Hence, $\dim\mathcal{A} \leq 18ks^2$. The algebra $\mathcal{A}$ contains the matrices $(M_1)^2$, $(M_3)^2$, $(M_1)^{-1}M_3 = M_2$ and $M_3(M_1)^{-1} = M_3M_2M_3$. Every matrix from the algebra $\mathcal{B}$ generated by $(M_1)^{-1}$ and M_3 (this is also the algebra generated by M_1, M_2 and M_3) is representable as $K + M_1L + M_3N$, $K, L, N \in \mathcal{A}$. Hence, $\dim\mathcal{B} \leq 54ks^2 < n^2 = 36k^2s^2$ and this cannot be $gl(n, \mathbf{C})$. By the Burnside theorem, $\mathcal{B}$ is reducible.

9 Proof of Theorem 15 in the case of matrices A_j

Suppose that the Deligne-Simpson problem is weakly solvable in one of cases A) – D) for matrices A_j with relatively generic but not generic eigenvalues. By Lemma 13 it is solvable as well.

Construct a Fuchsian system with matrices-residua from an irreducible triple or quadruple corresponding to one of the four cases and with relatively generic eigenvalues. One can multiply the matrices-residua by $c^* \in \mathbf{C}$ so that no two eigenvalues differ by a non-zero integer and the eigenvalues of the monodromy operators become relatively generic.

Hence, the monodromy group of the system is irreducible. Indeed, if it were reducible, then the eigenvalues of the diagonal blocks would satisfy only the basic non-genericity relation and its corollaries. The sum of the corresponding eigenvalues of the matrices-residua is 0 and, hence, one can conjugate simultaneously the matrices-residua to a block upper-triangular form, see [Bo2], Theorem 5.1.2. The irreducibility of the monodromy group contradicts part 1) of Theorem 15.

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